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The Complete Solution for Constrained Delay Problems in the Calculus of Variations by Unconstrained Methods

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This paper has two main ideas. The first idea is that general constrained problems with delay in the calculus of variations can be associated with unconstrained calculus of variations problems by using multipliers. This allows us to obtain a true Lagrange multiplier rule where the original variables, the multipliers, and the slack variables for inequality constraints can be determined. The second idea is that critical point solutions to the delay problem, which include the determination of the multipliers, immediately follow from Euler–Lagrange equations for the unconstrained problem. This critical point solution is a necessary condition for the original problem. In later work we will show that these methods can be combined with previous methods by the first author to obtain efficient and accurate numerical solutions to the original problem where no such general numerical methods currently seem to exist. This seems to be an important development since the lack of such methods has hindered the usefulness of the theory. © 1998 Academic Press

1. INTRODUCTION

The purpose of this paper is to give methods for the complete solution for constrained delay problems in the calculus of variations. This work is a continuation of [8, 1] which deal with the unconstrained problem. The necessary theory for this paper is the Bliss-type multiplier rule for delay problems given in [1].

Reference [1] also gives a brief history and justification for the importance of delay differential equations. Much of this can be found in Refs. [9–16].

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The remainder of this paper is as follows. In Section 2, we give our problem setting, the basic definitions and notations, and the multiplier rule is established [1]. In Section 3, we show that these necessary conditions are "equivalent" to the basic necessary conditions for a new unconstrained problem in the calculus of variations with delays, that is, to the condition that the first variation for the unconstrained problem vanishes for suitable test functions, which is in turn equivalent to the Euler–Lagrange equation, the corner conditions, and the transversality conditions.

Thus, we will have the analytic tools to find a solution to our problem when one exists. In addition, our new formulation will allow us to find the multipliers and will lead to efficient numerical procedures. In the event of inequality constraints, our method will also provide the amount of "slackness."

In Section 4, we give a nontrivial example to illustrate our theoretical results.

2. THE PROBLEM OF BOLZA WITH DELAY

The following problem will be considered in this paper:

$$\text{minimize} \qquad J(y) = \int_{t_1}^{t_2} f(t, y(t), y'(t), y(t-\tau), y'(t-\tau))\, dt \tag{1}$$

such that

$$\phi_\beta(t, y(t), y'(t), y(t-\tau), y'(t-\tau)) = 0, \qquad \beta = 1, \dots, m < n \tag{2}$$

$$\psi_\mu(t_1, y(t_1), t_2, y(t_2)) = 0, \qquad \mu = 1, \dots, p \leq 2n \tag{3}$$

and

$$y(t) = \alpha(t), \qquad t \in [t_1 - \tau, t_1] \tag{4}$$

where $t_1 < t_2$ are fixed in R; τ is a given positive real number such that $\tau < t_2 - t_1$; $y \in R^n$; $f : [t_1, t_2] \times (R^n)^4 \to R$; $\phi_\beta : [t_1, t_2] \times (R^n)^4 \to R$, $\beta = 1, \dots, m < n$; $\psi_\mu : [t_1, t_2]^2 \times (R^n)^2 \to R$, $\mu = 1, \dots, p \leq 2n$; and $\alpha : [t_1 - \tau, t_1] \to R^n$. Furthermore, f and ϕ_β are C^4 on their domain and y and α are piecewise smooth. This problem is essentially the same as The Problem of Bolza (see [1] or [2]) except that it contains time delay arguments. For this reason, we will refer to (1)–(4) as The Problem of Bolza with Time Delay Arguments and to an associated function $y(t)$ satisfying (2)–(4) as an admissible arc.

To simplify the notation and presentation, we suppress most of the arguments. Thus, the following definitions will be used throughout this paper:

$$f(t) \equiv f(t, y(t), y'(t), y(t-\tau), y'(t-\tau)),$$
$$y_\tau \equiv y(t-\tau),$$
$$y'_\tau \equiv y'(t-\tau),$$

and

$$\psi_\mu(y(t_1), y(t_2)) \equiv \psi_\mu(t_1, y(t_1), t_2, y(t_2)).$$

We also note that the equality constraint (2) can be replaced by an inequality constraint $\phi_\beta \le 0$ (as in Section 3).

The necessary solution to The Problem of Bolza with Time Delay Arguments is given in the following definition and Theorem 1 (see [1]).

DEFINITION. An admissible arc $y^*(t)$ is said to satisfy the *Multiplier Rule* if there exist constants l_0 and e_μ, not all zero, and a function

$$F(t, y, l_0, l_\beta) = l_0 f + \sum_{\beta=1}^{m} l_\beta \phi_\beta, \tag{5}$$

with multipliers l_β continuous on $[t_1, t_2]$, except possibly at corners of $y^*(t)$ such that the equations

$$\frac{d}{dt} F_{y'}(t) + \frac{d}{dt} F_{y'_\tau}(t+\tau) = F_y(t) + F_{y_\tau}(t+\tau), \qquad t_1 \le t \le t_2 - \tau \tag{6}$$

$$\frac{d}{dt} F_{y'}(t) = F_y(t), \qquad t_2 - \tau \le t \le t_2 \tag{7}$$

are satisfied along $y^*(t)$ in $[t_1, t_2]$ and furthermore, such that the equations

$$F_{y'}(t)\eta(t)\big|_{t_1}^{(t_2-\tau)^-} + F_{y'}(t)\eta(t)\big|_{(t_2-\tau)^+}^{t_2} + F_{y'_\tau}(t+\tau)\eta(t)\big|_{t_1}^{(t_2-\tau)^-} + e_\mu \Psi_\mu = 0 \tag{8}$$

$$\psi_\mu = 0 \tag{9}$$

hold along $y^*(t)$ for all admissible variations $\eta(t)$ and where repeated indices with respect to μ are summed.

Thus we have from [1],

THEOREM 1. *Every minimizing arc $y^*(t)$ must satisfy the multiplier rule.*

3. AN UNCONSTRAINED FORMULATION

The purpose of this section is to formulate an unconstrained problem associated with (1)–(4), above. For exposition purposes, we will give our basic results for equality constraints as in (2) and then describe the situation for inequality constraints, as in (2′), below.

For convenience, we assume $l_0 \neq 0$, in which case, we may choose $l_0 = 1$ and thus l_β will be uniquely determined. This is what Bliss calls the normal case (see [1]). We also assume that (3) and (4) hold; define $z_\beta(t)$ by

$$z'_\beta(t) = l_\beta(t), \qquad z_\beta(t_1) = 0;$$

$$Y(t) = \left(y(t)^T, z(t)^T\right)^T,$$

and

$$\begin{aligned} I(Y) &= \int_{t_1}^{t_2} \left[f(t, y(t), y'(t), y_\tau, y'_\tau) + z'_\beta(t)\phi_\beta(t, y(t), y'(t), y_\tau, y'_\tau)\right] dt \\ &= \int_{t_1}^{t_2} G(t, y(t), y'(t), y_\tau, y'_\tau)\, dt. \end{aligned} \tag{10}$$

In the above, Y is an $n + m$ dimensional vector and repeated indices in β are summed. Thus, (10) is an associated (or reformulated) unconstrained delay problem associated with (1)–(4).

We now show that the critical point condition, $I'(Y, W) = 0$ for admissible W in (10), is equivalent to the multiplier rule in Theorem 1. Thus, if $y(t)$ is a solution for The Problem of Bolza with Time Delay Arguments, then the Euler–Lagrange equations (11), the corner conditions (12), and the transversality conditions (13), given below, must hold. Furthermore if $y(t)$ is the unique solution, it is (formally) determined by (11)–(13).

On the interval $t_1 \leq t \leq t_2 - \tau$ we have the Euler–Lagrange equation

$$\frac{d}{dt}\left[G_{Y'}(t) + G_{Y'_\tau}(t + \tau)\right] = G_Y(t) + G_{Y_\tau}(t + \tau) \tag{11a}$$

and the condition that

$$G_{Y'}(t) + G_{Y'_\tau}(t + \tau) \text{ is continuous at corners of } Y. \tag{12a}$$

On the interval $t_2 - \tau \leq t \leq t_2$, we have the corresponding conditions

$$\frac{d}{dt} G_{Y'}(t) = G_Y(t) \tag{11b}$$

$$G_{Y'}(t) \text{ is continuous at corners of } Y, \tag{12b}$$

and the transversality condition

$$G_{\bar{Y}'}(t_2) = 0. \tag{13}$$

In addition, we require

$$\left[G_{Y'}(t) + G_{Y'_\tau}(t+\tau)\right]\big|_{(t_2-\tau)^-} = G_{Y'}(t)\big|_{(t_2-\tau)^+}. \tag{12c}$$

In the above $G_{Y'_\tau}(t+\tau)$ indicates, for example, differentiation with respect to $Y'_\tau = (y'^T_\tau, 0^T)^T$ with arguments at $t+\tau$. In (13) below, $\bar{Y}$ is the vector obtained from Y where $Y_i(t_2)$, the ith component of Y, is not specified. Thus, $\bar{Y}$ has at least m components since $z(t_2)$ is not specified.

To clarify our ideas we note that in (11a)

$$G_{Y'}(t) + G_{Y'_\tau}(t+\tau) = \begin{pmatrix} f_{y'} + z'_\beta \phi_{\beta y'} + f_{y'_\tau}(t+\tau) + z'_\beta \phi_{\beta y'_\tau}(t+\tau) \\ \phi \end{pmatrix}$$

and

$$G_Y(t) + G_{Y'_\tau} = \begin{pmatrix} f_{y'} + z'_\beta \phi_{\beta y} + f_{y_\tau}(t+\tau) + z'_\beta \phi_{\beta y_\tau}(t+\tau) \\ 0 \end{pmatrix} \tag{14}$$

are $n+m$ vectors and $\phi_{\beta y'}$ is the derivative of ϕ_β with respect to y' (for example). Now, the last m components of (11a) imply that $\phi'(t) = 0$ and hence $\phi(t)$ is constant between corners on $t_1 < t < t_2 - \tau$ and similarly, for $t_2 - \tau < t < t_2$ by (11b). The last m components of the corner conditions (12) imply that all constants are equal, and the transversality condition (13) implies that $\phi(t) \equiv 0$ in $[t_1, t_2]$.

The first n conditions of (11)–(13) yield the other necessary conditions in the multiplier rule. Thus, we have proven

THEOREM 2. *If* $Y = (y^T, z^T)^T$ *satisfies the critical point conditions* (11)–(13) *for the unconstrained problem* (10), *then y satisfies the multiplier rule with* $z'_\beta = l_\beta$.

In the event of an inequality constraint in (2), for example, $\bar{\phi}_{\beta_0} \leq 0$, we set

$$\phi_{\beta_0} = \bar{\phi}_{\beta_0} + w'^2(t) \equiv 0 \tag{2'}$$

as one of the equalities in (2), where β_0 is a fixed positive integer less than m, and w' represents a "slack" variable. In this case we define $Y(t) = (y(t)^T, z(t)^T, z(t)^T, w(t)^T)^T$ and proceed as above. The only change is that $G_{Y'}$ in (14) is now an $n+m+1$ vector and, proceeding as in our proof of the case where $\phi = 0$, above, we have

$$z_{\beta_0}(t)w(t) \equiv 0 \qquad \text{on}[t_1, t_2]. \tag{15}$$

This result is the expected inequality result that at each point t, either the multiplier $z_{\beta_0}(t)$ or the associated slack variable $w(t)$ vanishes.

COROLLARY 3. *If $\phi_{\beta_0}(t) \leq 0$ for some β_0 in* (2′), *then in addition to the results in Theorem* 2, *we have* (15).

If there are $m' \leq m$ inequalities then Y is now an $n + m + m'$ vector. Specifically, (14) becomes the $n + m + m'$ vector

$$G_{Y'}(t) + G_{Y'_\tau}(t+\tau) = \begin{pmatrix} f_{y'} + z'_\beta \phi_{\beta y'} + f_{y'_\tau}(t+\tau) + z'_\beta \phi_{\beta y'_\tau}(t+\tau) \\ \phi \\ 2 z_\beta w_{\beta'} \end{pmatrix},$$

where $\beta' = 1, 2, \ldots, m'$ corresponds to the indices for the inequality constraints and

$$G_Y(t) + G_{Y'_\tau} = \begin{pmatrix} f_{y'} + z'_\beta \phi_{\beta y} + f_{y_\tau}(t+\tau) + z'_\beta \phi_{\beta y_\tau}(t+\tau) \\ 0 \\ 0 \end{pmatrix}.$$

4. AN EXAMPLE

As an example of the theory given above, we consider the following constrained variational problem with time delay,

$$\text{minimize} \qquad J(y) = \frac{1}{2}\int_0^2 y_2'^2(t)\,dt,$$

such that

$$y_1'(t) + y_1(t-1) - y_2'(t) = 0,$$

with

$$y_1(t) = 1, \qquad t \in [-1, 0].$$

This example arises in minimum energy control of a time delay system. For this example, function G is given as

$$G = \tfrac{1}{2} y_2'^2(t) + z'(t)\left[y_1'(t) + y_1(t-1) - y_2'(t)\right].$$

It can be shown that

$$y_1(t) = \begin{cases} \frac{3}{16}t^2 - t + 1, & 0 \le t \le 1 \\ -\frac{1}{16}(t-1)^3 + \frac{1}{2}(t-1)^2 - \frac{5}{8}t + \frac{13}{16}, & 1 \le t \le 2, \end{cases}$$

$$y_2(t) = z(t) = \begin{cases} \frac{3}{16}t^2, & 0 \le t \le 1 \\ \frac{3}{8}t - \frac{3}{16}, & 1 \le t \le 2, \end{cases} \tag{16}$$

satisfy the Euler−Lagrange equations given in (6) and (7) as well as the corner condition at $t = 1$ given in (12c) and the given constrained equation and boundary conditions.

To obtain these results we define, as indicated above,

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ z(t) \end{pmatrix}, \qquad z(0) = 0,$$

and

$$I(Y) = \int_0^2 \left\{ \frac{1}{2} y_2'^2(t) + z'(t)\left[y_1'(t) + y_1(t-1) - y_2(t) \right] \right\} dt.$$

Thus, from (11a), we have on $0 \le t \le 1$

$$\frac{d}{dt}\begin{pmatrix} z'(t) \\ y_2'(t) - z'(t) \\ y_1'(t) + y_1(t-1) - y_2'(t) \end{pmatrix} = \begin{pmatrix} z'(t+1) \\ 0 \\ 0 \end{pmatrix}$$

and the reader may verify that (16) satisfies the above equality.

On $1 \le t \le 2$ we have, from (11b),

$$\frac{d}{dt}\begin{pmatrix} z'(t) \\ y_2'(t) - z'(t) \\ y_1'(t) + y_1(t-1) - y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and the reader may also verify that (16) satifies the above equality.

At $t = 1$ we have

$$\begin{pmatrix} z'(t) \\ y_2'(t) - z'(t) \\ y_1'(t) + y_1(t-1) - y_2'(t) \end{pmatrix}\Bigg|_{1^-} = \begin{pmatrix} z'(t) \\ y_2'(t) - z'(t) \\ y_1'(t) + y_1(t-1) - y_2'(t) \end{pmatrix}\Bigg|_{1^+}$$

and (16) holds once again.

Finally, $\overline{Y} = \binom{y_z}{z}$ so that (13) becomes

$$\begin{pmatrix} y_2'(t) - z'(t) \\ y_1'(t) + y_1(t-1) - y_2'(t) \end{pmatrix}\Bigg|_{t=2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the reader may verify that (16) satifies the above equality.

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